3rd Australasian Engineering Heritage Conference 2009

Interactive Analysis of Arching Masonry Structures

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SUMMARY: *In the modern world, it is often assumed that every structure can be analysed simply to provide thorough understanding of behaviour. Modern computer analyses, however, frequently yield results that are clearly in error, for example, indicating collapse of structures that are behaving well. The paper discusses a range of specific problems with historic masonry structures where expensive finite element (FE) analysis led to unacceptable results and much simpler interactive equilibrium studies showed the structure to be sound.*

The paper is illustrated with bespoke spreadsheet graphical analyses of specific complex vaulted and arching structures.

1 INTRODUCTION

Arches have been used to span openings for at least 3000 years. Throughout most of that time, they were designed by rule of thumb, geometric rules developed and handed down from master to apprentice. The rules were carefully guarded and to a large extent have only become known through back analysis of ancient structures.

When Wren and Hooke were rebuilding London after the fire of 1666 they found it necessary, for the first time, to develop reasonable estimates of the abutment forces and buttresses required. Hooke (1) realised that arches and chains were just mirror images. The concept of the thrust line was born

By 1846, Barlow (2) had demonstrated that we cannot know the true thrust line but that we could be certain of the stability of the structure. This was perhaps the first formulation of the plastic theorems, so it is interesting that the last 50 years have seen a huge argument over whether they can apply to masonry, a brittle material.

The author began work on arches in 1981 by which time Heyman (3) had been publishing in academic journals for 15 years and more but his paper of 1980 in the ICE Proceedings provided an interesting basis for development.

Through the 1980s, graphical analysis was possible on computers, but only using established programming languages. The program Archie was developed at the University of Dundee and became popular in the sphere of bridge assessment in the UK. By the end of that decade spreadsheets were becoming popular and the new formulations developed in programming arch analysis were ideally suited to the new form.

2 SPREADSHEET CONSTRAINTS

Spreadsheets are ideally suited to tabular computation. It happened that the form of calculation developed within Archie is also ideally suited to tabular work. I will begin with some advice on spreadsheet use from accumulated practice.

2.1 Core calculations

Spreadsheets are computer programs like any other and subject to similar errors. In the battle to produce error-free code it is necessary to employ well-established tactics. Perhaps the most important is never to write a long formula in a single cell. There are thousands of cells available, they cost nothing, calculations are just as well done in small steps.

Great care should be taken in developing calculations so that addressing is clear and unambiguous and that each calculation can be replicated for the next element of the structure without any retyping.

2.2 Graphing

The facilities for graphing within Excel spreadsheet program are extremely powerful. It is possible to produce output that looks very much like an engineering drawing. Perhaps the most important capacity is the ability to lift the pen by simply missing one line in a column of plotted points. This can also be achieved conditionally by putting “#N/A” as text in a cell. Attempts to blank the cell using “” are seem to leave something not recognised as a blank by the graphing routines.

2.3 Choices

When programming, it is often necessary to make choices. The IF statement is very powerful but also very easy to miss-program. It is much more secure to make a truth table, for example by saying =A>B, and then use the outcome within an IF statement.

	A	B	C	D
7	=A5<B5	=B5<=C5	=AND(A7,B7)	=IF(C7,

***Table 1** One row of a Truth Table*

3 CALCULATION PROCESSES

3.1 The meaning of the thrust line

The thrust line forms the locus of the centroid of compressive stress as it flows from section to section. Thrust lines only work in the essentially skeletal structures. The results are sensitive to the choice of divisions between the sections as will be illustrated below.

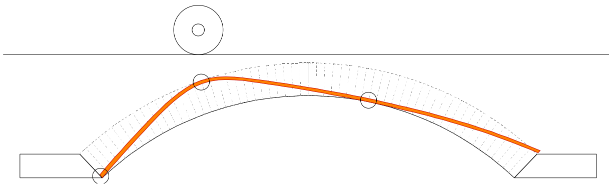

***Figure 1** A thrust line for an arch bridge*

The simplest view of the calculation process is that it relates directly to computing the stability of the masonry retaining wall. Moments are taken about a specific point and divided by the force normal to the section. In circular curves it is convenient to take moments about the centre of the circle. If the curve is more complex the scheme of axes must be developed specifically and moved for each segment.

3.2 Vector format calculations

In Britain, at least, it remains uncommon for Civil Engineering students to be taught to use vectors in their calculations. In two dimensions, the vector format offers an number of simplifications in the algebraic side of calculations. Once a move is made to three dimensions the benefits become substantial. It is perhaps appropriate to introduce the calculation methods used in the work described below. Some of the steps are very simple.

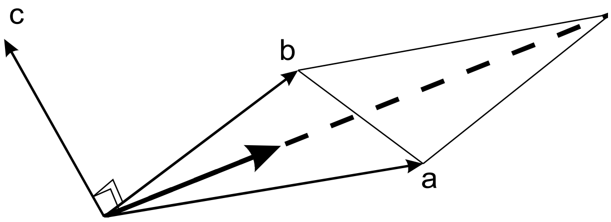

***Figure 2** Area and centroid of a triangle*

$$c = \begin{vmatrix} i & j & k \\ a_x & a_y & 0 \\ b_x & b_y & 0 \end{vmatrix}$$

The area of a parallelogram bounded by two vectors is the cross product of the vectors. It is a vector quantity, having direction normal to the plane of the two vectors. The direction of the vector product depends on the order of multiplication so that a random polygon area can be described by the vectors to the corners and the area of the polygon found by taking the cross products of the vectors in sequence. The signs of various partial areas take care of themselves.

In Figure 2, **a** x **b** is the area of the parallelogram and is the vector **c** The area of the triangle is thus |**c**|/2. If the product is taken as **b** x **a** the product is –**c**. The centroid of the triangle is at (**a** + **b**)/3 Strictly, of course, this is (**a** + **b** + **0**)/3, the three vectors describing the three corners.

In three dimensions the equivalent form is the tetrahedron.

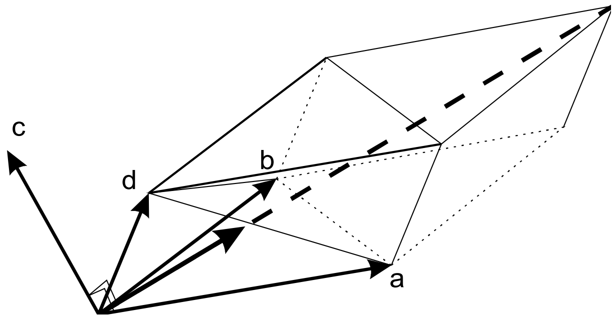

***Figure 3** Volume and centroid of a tetrahedron*

In Figure 3 the vector **d** describes the third dimension of a parallelepiped. The scalar product **c . d** is the product of the magnitude of **c** and the projection of **d** on **c**. Which is the volume of the parallelepiped. The centroid is at (**a** + **b** + **d**)/4.

Perhaps of more significance is the representation of forces as vectors. A force in three dimensional space made up of three components Fx, Fy, Fz is not completely described as the position of the force is significant. If we choose a spatial origin such that the force acts at a point x,y,z, then the position of the force can be defined by its moment about the origin. The moment about the origin is the cross product of the vector defining the force and the geometric vector from the origin to any point on the line of action of the force.

$$M = \begin{vmatrix} i & j & k \\ F_x & F_y & F_z \\ x & y & z \end{vmatrix}$$

Two forces defined in this way can be added by simply adding the six components Fx, Fy, Fz, Mx, My, Mz.

It is perhaps becoming clear that this form of calculation lends itself to tabular presentation and thus to calculation in a spreadsheet.

3.3 *Three dimensional thrust diagrams and the Wrench*

Robert Hooke (1) conceived the thrust line while he and Christopher Wren were rebuilding London's Churches after the great fire. It is a very powerful tool but has some very strict limitations. The first of these is that it only works in skeletal structures. There is very limited value (and great opportunity for confusion) in drawing thrust lines through a continuum. This is perhaps best illustrated in a domain where two thrust lines meet, something that Brunel dealt with in his analysis of the Maidenhead bridge

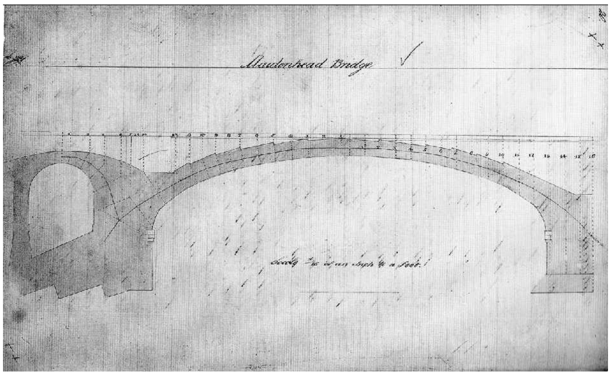

Figure 4 *Brunel's trust lines for Maidenhead*

The bifurcated thrust at the left of the main arch in Figure 4 actually tells us nothing. The flow of force is entirely dependent on the lines on which the structure is divided. This is well illustrated by considering thrusts in a vault in a Scottish tower house. The vault is semi-circular and 5m span with walls nearly 1m thick, so the material through which the thrust is to be plotted is essentially a continuum rather than a skeleton.

A spreadsheet was built in which the structure was divided radially into segments but the centre of the divisions could be moved down from the centre of the semi-circle to a point up to 50m lower. In this latter scheme, the divisions are approaching vertical slices.

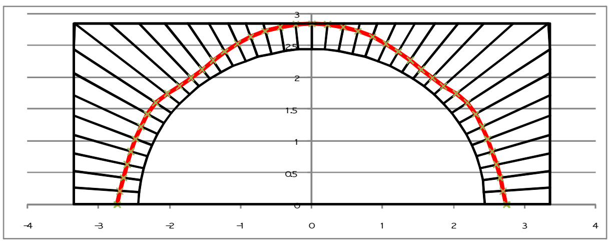

Figure 5 *Thrust line through a substantial structure divided radially*

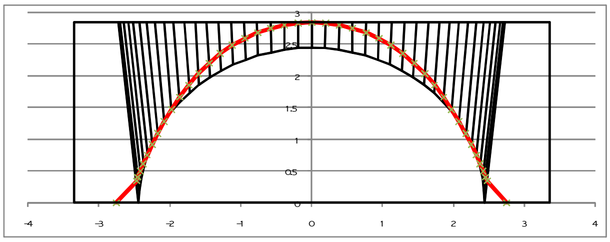

Figure 6 *Thrust line through a substantial structure divided almost vertically*

Note that in both Figure 5 and Figure 6 the thrust begins and ends, and passes through the crown at the same point. The overall stability is not changed, it is simply that it becomes impossible to trace the thrust through the sections. In arch bridges, this is overcome by the often false assumption that the arch is the entire structure and everything else acts upon it as load or reaction.

In the discussion so far, the issue has been entirely in two dimensions. If we make an arbitrary cut through a three dimensional structure, we have seen that the forces acting across the cut can be represented by three components of force and three of moment. WE may choose a local set of axes so that two lie in the cut plane and one normal to it. We then have a normal component and two shear forces, moments about two axes in the plane, which can be represented by placing the normal force in an appropriate position, but there remains a moment about the normal axis. In those few areas of structural teaching where this problem is discussed, the general form of a force is called a wrench. That is a force in a particular position in a specified direction but with an associated twisting moment about the line of action. The first example below is of a skeletal stone structure in which the wrench cannot reasonably be ignored.

3.4 *Interaction and its validity*

In 1846, WH Barlow (2) set out his views of the way arches behaved. He made a practical (Figure 7) demonstration of the fact that the thrust in an arch might take any one of many lines and that the engineer could not expect to show which one was "correct". This is perhaps the earliest expression of the plastic theorems which didn't otherwise appear in the UK until Baker (5) returned from the IABSE symposium in 1936.

***Figure 7** A replica of Barlow's model showing how unpredictable a thrust line is*

Half a century later, Castigliano (6) developed a scheme for the elastic analysis of indeterminate structures. Using hand calculation he used this system to analyse masonry arches, progressively removing (or replacing) any material that went into or came out of tension as the iterative process proceeded. He recognised, though, that his analysis was critically dependent on the boundary conditions and that in the limit, the analysis produced a mechanism at failure which could be arrived at directly with much less effort.

In many circumstances, it is useful to explore the range of thrust lines that might be appropriate in a structure. The easiest way to do this is with interaction. Allow the engineer to vary parameters and explore the effects. This is a much more powerful process than might first appear. Certainly much more so than computer optimisation of structures with which a direct comparison might be made. The engineer maintains direct and continuous contact with the exploration of equilibrium.

Modern spreadsheets offer very powerful tools for building interactive analyses. Scroll bars, spin buttons, radio buttons and check boxes all have their place. Figure 8 shows a range of tools in use. Placing radio buttons in a box automatically connects them so that it is possible to choose a shape and the end conditions for an arch independently.

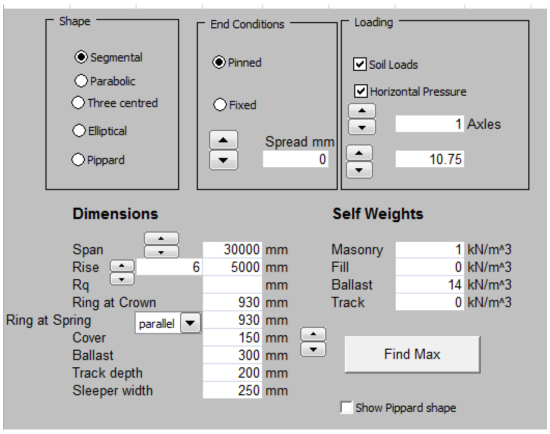

***Figure 8** A selection of tools used to add interaction to a spreadsheet*

4 EXAMPLES

4.1 Wells Cathedral

In the cathedral at Wells there is a flying buttress that has been seriously modified. Where it used to reach the ground as a substantial masonry pier it now lands on a vaulted ceiling with a small column below. To complicate the issue the column is not central under the buttress. The buttress therefore slopes sideways to land on the centre of the pier (Figure 9).

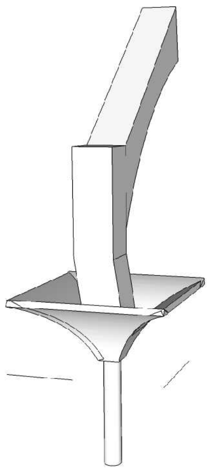

***Figure 9** View of the buttress and vaults*

The model was constructed in Excel. The buttress, vaults and column have a dominantly vertical aspect so they could be divided into horizontal slices. Each slice was represented as an octagon. The volume was built up from a set of tetrahedra for which both volume and centroid are easily calculated. Sections which did not require an octagonal model were represented by making some nodes co-incident.

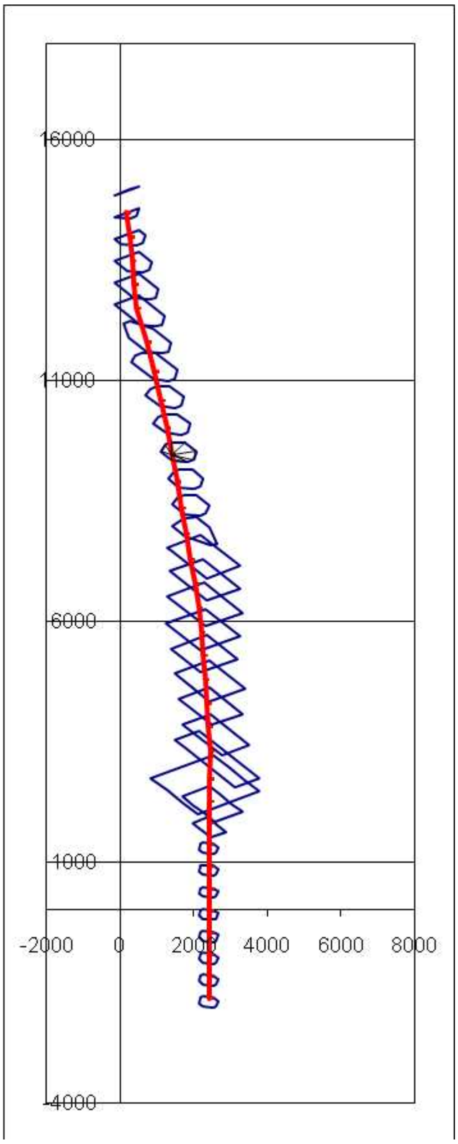

Figure 10 *A 3D view of thrust in the Wells Buttress*

This model illustrates a number of features available in Excel. The buttress itself is only part of the structure. Forces are applied in the top and transmitted through it to the ground. The model allows the engineering to apply appropriate forces at the top of the buttress and computes the flow and then displays it in several ways. The two-dimensional graph is a representation of a three-dimensional structure and using scrollbars it is possible to rotate the image about each axis.

Despite this ease of manipulation, it remains very difficult, if not impossible to visualise the flow of force through the structure in this form. In Figure 10, one slice contains a spider linking the corner nodes with the appropriate node on the thrust. This helps a little, but a direct cross section is also needed.

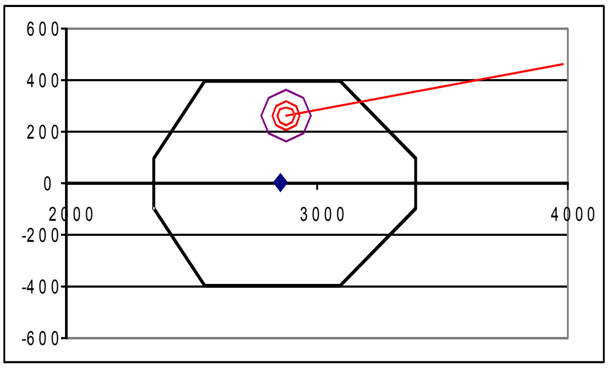

Figure 11 *A plan view of the selected slice from figure 8*

The section shown in Figure 11 shows the position of the centroid of the section and the point of action of the normal force. The in plane forces are represented by a vector showing magnitude and direction. Three rings surround the point of the force. The inner circle contains an area big enough to sustain the whole normal force. The thin annulus around it is capable alone of resisting the shear forces. The final ring, if it contained shear force circling the centre, would be capable of resisting the torsion on this face. Thus, this diagram is capable of showing that the structure is (at this section) able to sustain the full system of forces applied.

4.2 Removing a span of an arch bridge

There are many arch bridges on the UK railways. Occasionally it is necessary to modify one span. In this case the bridge was highly skewed and the central span was made up from separate ribs. A side span was to be removed and replaced with beams. It was necessary to demonstrate first that the main span could stand without the side and then that once the beams were in place the bridge could function properly.

An initial model was made using the Archie program. In this model the beam span was modelled as a flat arch. This produced an accurate representation of the weight of the span but with a substantial thrust.

The output from Archie was then fed into an Excel spreadsheet (Figure 12) for further manipulation. In the spreadsheet model it was possible to remove the thrust from the side span and to move the reaction from the beams to find the best combination of span weight and support.

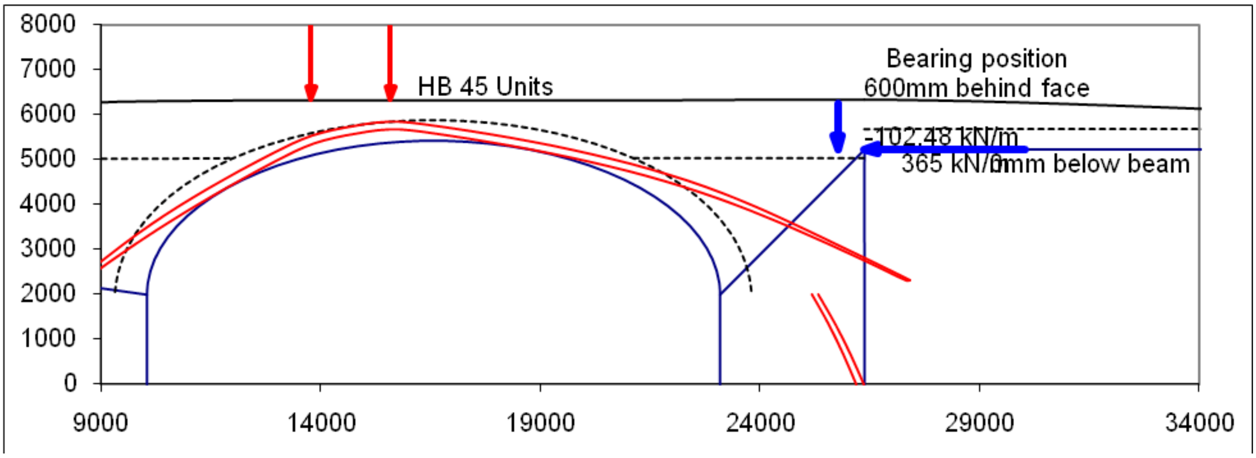

Figure 12 *Thrust layout in a bridge with a beam span, plotted in Excel*

4.3 Clerkenwell house of detention

The third example is a very complex structure. It is the basement floor of a disused and demolished prison. It was built in the late 1700s. The space above is now a car park and is accessed by heavy lorries. The floor was not designed to carry such loads but has been doing so for some years. To bring the basement into use as an office it has been necessary to analyse it for the ability to carry heavier live loads.

The most complex part of the structure consists of a central barrel vault of 4m span and 0.6m rise which is supported at each edge on the wall which is in turn pierced by arches. Behind the wall is a row of 2.1m span vaults spanning in the opposite direction.

Figure 13 *General view of Clerkenwell vaults*

These vaults are also supported on dividing walls which in turn are pierced by arches. The whole area is of some 9m x 7m. The vaults are one brick thick, the walls one and a half brick thick and the arches are supported at the intersection on granite columns which are octagonal and 330 mm wide. Figure 13 shows a view through the side arches into the main area. Figure 14 is a plan view of the critical area showing the layout of vaults and possible loads.

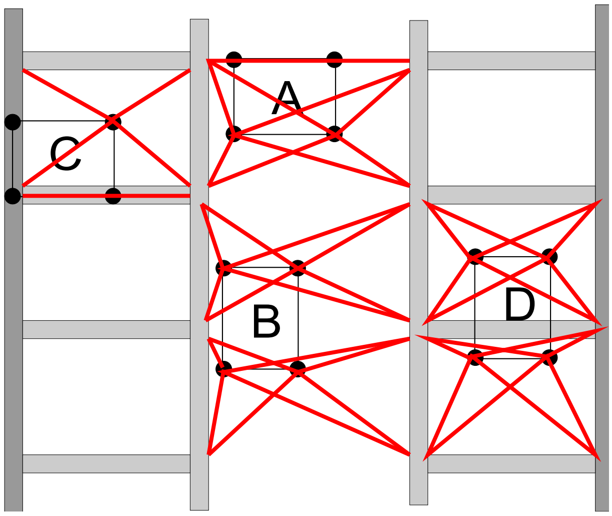

Figure 14 *Plan view of vaults showing load paths*

The analysis considered the possibility of wheel loads over stressing the vaults, of the vaults overturning their abutments and conversely of the archers exerting too great a load on the edge of the vaults causing them to fail.

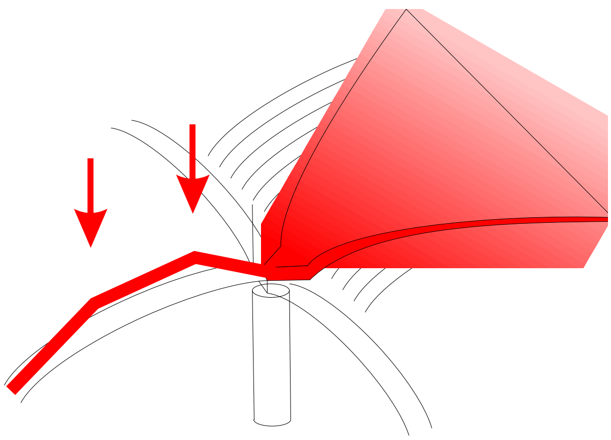

Figure 15 *Side arch load transmitted to vault*

In order to complete this analysis it was a necessary first to envisage a pattern of distribution of stress within the vaults. The model is complex but still works well in

Excel. For speed of development it was necessary to keep several parts of the model essentially separate and pass data either manually or by linking spreadsheets.

Figure 15 illustrates how the thrust from the side arch might be distributed in the main vault. Figure 16 illustrates transmission through the long wall where a high narrow stress zone must be converted to a low wide one

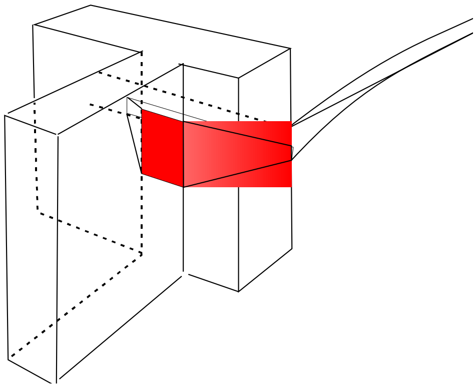

Figure 16 *Thrust transmission through wall.*

Figure 17 shows the effect of transverse distribution (Figure 15) on the thrust pattern in the vault.

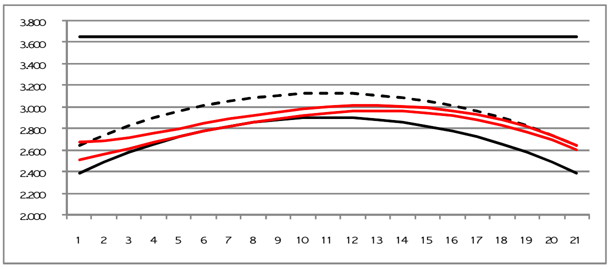

Figure 17 *Effect of distribution of side thrust into main vault*

Figure 18 shows the effect on the side arch of a large load in the main span. Dealing with the reaction to this force on the outer wall is, perhaps, the biggest issue in this work.

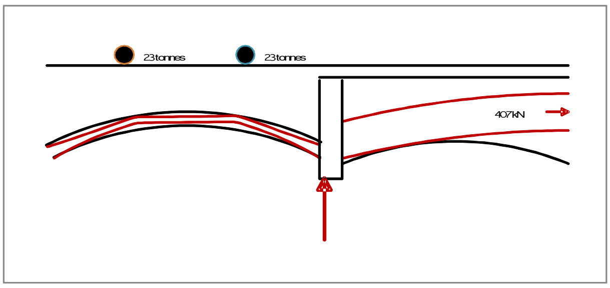

Figure 18 *Transmission of main vault thrust into side arch*

The outcome of the work was to show that the structure is indeed capable of carrying the large loads required but not with the factors of safety one would expect if the building would be occupied.

5 CONCLUSIONS

A review of three projects has shown how it is possible to analyse very complex structures using simple procedures and a powerful spreadsheet. Some suggestions have been made as to how spreadsheets should be structured and how output can be developed. The interactive nature of the procedures is quite different from normal modern analysis. However, it has the considerable advantage of keeping the engineer closely involved with the process.

What has not been made clear, though it is undoubtedly true, is that the powerful modern tools of finite elements and discrete elements, though apparently able to produce definitive analyses of such structures, actually rarely deliver what is promised. The modelling process is extremely complex and requires very special skills. Some of the boundary conditions are quite unknown, and the approximations used are usually inadequate. The most common result from an expensive analysis is an indication that the structure is severely overstressed in certain locations.

6 REFERENCES

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